

Magnetars oscillations in the presence of a crust

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30 May 2011

ABSTRACT

We study axisymmetric perturbations of neutron star endowed with a strong magnetic field (magnetars), considering the coupled oscillations of the fluid core with the solid crust. We recover discrete oscillations based mainly in the crust and a continuum in the core. We also confirm the presence of “discrete Alfvén modes” in the gap between two contiguous continua (see van Hoven & Levin (2011)) and, in addition, we can resolve some of them also inside the continua. Our results can explain both the lower and the higher observed quasi periodical oscillations (QPOs) in SGR 1806-20 and SGR 1900+14 and put constraints on the mass, radius and crust thickness of the two magnetars.

Key words: asteroseismology – MHD – stars: neutron – stars: oscillations – stars: magnetic fields – gamma rays: general

1 INTRODUCTION

Over the last few years, a number of observational discoveries have brought magnetars (ultra-magnetized isolated neutron stars) to the forefront of researchers attention. These extreme objects comprise the Anomalous X-ray Pulsars (AXPs; 10 objects) and the Soft Gamma-ray Repeaters (SGRs; 5 objects), which are observationally very similar classes in many respects. They are all slowly rotating X-ray pulsars with spin periods clustered in a narrow range ($P \sim 2\text{--}12\text{ s}$), relatively large period derivatives ($\dot{P} \sim 10^{-13} - 10^{-10}\text{ s s}^{-1}$), spin-down ages of $10^3 - 10^4\text{ yr}$, and magnetic fields, as inferred from the classical magnetic dipole spin-down formula, of $10^{14} - 10^{15}\text{ G}$ (for a recent review see Mereghetti (2008)).

SGRs undergo periods of activity during which recurrent bursts with sub-second duration and peak luminosities of $\sim 10^{38} - 10^{41}\text{ erg/s}$ are observed. SGRs also show, on rare occasions, extreme events known as giant flares. These are characterized by an initial spike of duration comparable to that of recurrent bursts, but many orders of magnitude higher luminosity. Only three giant flares have so far been observed in over 30 yr of monitoring.

According to the magnetar model (Thompson & Duncan 1993, 2001) energy is fed impulsively to the neutron star magnetosphere when local “crustquakes” let magnetic helicity propagate outwards, giving rise to recurrent bursts with a large range of amplitudes. Giant flares are believed to orig-

inate from large-scale rearrangements of the inner field or catastrophic instabilities in the magnetosphere (Thompson & Duncan 2001; Lyutikov 2003). Most of this energy breaks out of the magnetosphere in a fireball of plasma expanding at relativistic speeds which results in the initial spike of giant flares. The decaying, oscillating tail that follows the spike displays many tens of cycles at the neutron star spin rate. This is interpreted as being due to a “trapped fireball” which remains anchored inside the magnetosphere and cools down in a few minutes. The total energy released in this tail is $\sim 10^{44}\text{ erg}$ in all three events detected so far.

A power spectrum analysis of the high time resolution data from the 2004 Dec 27 event of SGR1806-20, observed with the X-Ray Timing Explorer (RXTE), led to the discovery of fast Quasi Periodic Oscillations (QPOs) in the X-ray flux of the decaying tail of SGR (Israel et al. (2005)). QPOs with different frequencies were detected, some of which were active simultaneously and displayed highly significant QPO signals at about 18, 26, 30, 93, 150, 625 and 1840 Hz (Watts & Strohmayer 2006). A re-analysis of the decaying tail data from the 1998 giant flare of another magnetar, SGR 1900+14, revealed QPOs around frequencies of 28, 54, 84 and 155 Hz (Strohmayer & Watts 2006). Hints for a signal at $\sim 43\text{ Hz}$ in the March 1979 event from SGR 0526-66 were reported as early as 1983 (Barat et al. 1983). All QPO signals show large amplitude variations with time and especially with the phase of the stars rotational modulation.

QPOs have also been argued to provide independent evidence for superstrong magnetic fields in SGRs Vietri et al. (2007). Numerous explanations have been proposed for the

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origin of the QPOs including the torsional oscillations of the crust alone or even as global seismic vibration modes of magnetars (Piro 2005; Sotani et al. 2007a; Samuelsson & Andersson 2007; Steiner & Watts 2009). Moreover, Levin (2006) argued that the QPOs may be driven by the global mode of the magneto-hydrodynamic (MHD) fluid core of the neutron star and its crust, rather than the mechanical mode of the crust. Following this idea, Sotani et al. (2008); Colaiuda et al. (2009); Cerdá-Durán et al. (2009) recently made two-dimensional numerical simulations (both linear and non-linear) and found that the Alfvén oscillations form continua which may explain the observed QPOs. The weak point of these simulations is the absence of the crust. Moreover, some of the expected eigenfrequencies (Duncan 1998; Israel et al. 2005; Piro 2005; Sotani et al. 2007a, 2008; Colaiuda et al. 2009; Cerdá-Durán et al. 2009) match the observed QPO frequencies found by Strohmayer & Watts (2006). These findings have opened the field of magnetar seismology, which provides an unprecedented probe of the star’s crust and interior. In retrospective, predictions made in Colaiuda et al. (2009) led to re-analysis of the 2004 Dec 27 event and revealed some new frequencies (Hambaryan et al. 2011).

In this work we move to a more general model for neutron star i.e. we assume a fluid interior and a solid crust which are threatened by a dipole magnetic field. In the interior the perturbations will be dominated by Alfvén waves propagating along the field lines forming a continuum spectrum as it has been seen in (Sotani et al. 2008; Colaiuda et al. 2009; Cerdá-Durán et al. 2009) while in the crust torsional oscillations dominate (Sotani et al. 2007a). Our results differ partially from the ones by Gabler et al. (2011), where the authors study a magnetised neutron stars with the presence of an elastic crust: in fact, we discover the presence of a type of “discrete Alfvén modes” which are not present in the absence of a crust. These modes have been seen by van Hoven & Levin (2011) in the gaps between two contiguous continua (for this reason they call them also “gaps modes”). However, we are able to resolve these modes also inside the continua.

2 DESCRIPTION OF THE PROBLEM

Here we give a brief description of the equations and the boundary conditions that we used in our study. First, we consider a spherically symmetric and static star, described by the TOV equations and the line element

$$ds^2 = -e^{2\Phi(r)} dt^2 + e^{2\Lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1)$$

Although here we used an ultra strong magnetic field $B = 4 \times 10^{15}$ Gauss, we neglect the induced deformation due to magnetic pressure, since it is still small because the magnetic energy E_m is a few orders of magnitude smaller than the gravitational binding energy E_g . Typically, $E_m/E_g \simeq 10^{-4}(B/10^{16}G)^2$, see Colaiuda et al. (2008) and Haskell et al. (2008) for results on magnetar’s deformation. Axial and polar perturbations don’t couple to each other and could be evolved independently, because we consider pure axisymmetric perturbations on a spherically symmetric star. Note that this assumption is not valid in a non-axisymmetric background: for example, as we mentioned be-

fore, a strong magnetic field will slightly deform the star away from sphericity and in this case there is a coupling of axial and polar oscillations. We have chosen to ignore the effects of smaller deformations since although they will introduce a tiny coupling between the polar and axial modes, in practice the properties of the two spectra will not be affected neither qualitative or quantitative. Other effects, such as resistivity, can potentially have more significant contributions (see Lander & Jones (2010)).

In the following we consider only torsional oscillations, which are of axial type and do not induce density variations in spherical stars: in this way, the radiative part of the metric describing the gravitational field does not vary significantly. For this reason the frequency of torsional oscillations is determined with satisfactory accuracy even when the metric perturbations were completely ignored by setting $\delta g_{\mu\nu} = 0$, i.e. working in the Cowling approximation (Cowling (1941)). In Sotani et al. (2007a) the MHD oscillations of the equilibrium model were derived by transforming the perturbed linearized equation of motion and the perturbed magnetic induction equations into 2D-wave equation for the displacement function $\mathcal{Y}(t, r, \theta)$ which is related to the contravariant component of the perturbed 4-velocity, δu^ϕ via

$$\delta u^\phi = e^{-\Phi} \partial_t \mathcal{Y}(t, r, \theta); \quad (2)$$

see Sotani et al. (2007a) for an analytical derivation. The 2D-wave equation for the displacement $\mathcal{Y}(t, r, \theta)$ has the following form:

$$A_{tt} \frac{\partial^2 \mathcal{Y}}{\partial t^2} = A_{20} \frac{\partial^2 \mathcal{Y}}{\partial r^2} + A_{11} \frac{\partial^2 \mathcal{Y}}{\partial r \partial \theta} + A_{02} \frac{\partial^2 \mathcal{Y}}{\partial \theta^2} + A_{10} \frac{\partial \mathcal{Y}}{\partial r} + A_{01} \frac{\partial \mathcal{Y}}{\partial \theta}. \quad (3)$$

All coefficients $A_{tt}, A_{20}, A_{11}, A_{02}, A_{10}$ and A_{01} depend only on r and θ , see Sotani et al. (2008). In Colaiuda et al. (2009) a coordinate transformation of the form

$$X = \pm \sqrt{a_1} \sin \theta \quad \text{and} \quad Y = \pm \sqrt{a_1} \cos \theta, \quad (4)$$

is used to transform the 2D-wave equations into a 1D-wave equation in the part of the star where the shear modulus was zero (core). The final form of this equation is:

$$A_{tt} \frac{\partial^2 \mathcal{Y}}{\partial t^2} = \tilde{A}_{02} \frac{\partial^2 \mathcal{Y}}{\partial Y^2} + \tilde{A}_{01} \frac{\partial \mathcal{Y}}{\partial Y}, \quad (5)$$

where

$$\tilde{A}_{02} = \frac{1}{4\pi r^4} a_1 a_1'^2, \quad (6)$$

$$\tilde{A}_{01} = \frac{X}{2\pi r^4} a_1 \left(\frac{2}{r^2} a_1 e^{2\Lambda} - 4\pi j_1 e^{2\Lambda} - \frac{a_1'^2}{2} \right). \quad (7)$$

Here $a_1(r)$ and $j_1(r)$ are the radial components of the electromagnetic 4-potential and the 4-current and a prime indicates derivative with respect the radial coordinate, respectively. Note that the function $a_1(r)$ is dimensionless.

The distribution of the magnetic field inside the star can be found by solving the Grad-Shavranov equation (Grad & Rubin (1958), Shafranov (1966)):

$$a_1'' e^{-2\Lambda} + (\Phi' - \Lambda') e^{-2\Lambda} a_1' - \frac{2}{r^2} a_1 = -4\pi j_1, \quad (8)$$

with the appropriate boundary and initial conditions, see Colaiuda et al. (2008) for details. As discussed in Colaiuda et al. (2009), $a_1(r)$ shows a maximum inside the star and then the transformation (4) is not any more one to one.

Therefore, after this maximum we choose the minus sign in equation (4).

In this paper we extend our previous work incorporating a solid crust in the magnetar model. The presence of the crust is accompanied with a non-zero shear modulus μ in the coefficients of the wave equation (3). The presence of the shear modulus μ does not allow for dimensional reduction and the transformation of equation (3) leads again into 2D-wave equation, in the new coordinates X and Y . The new equation has the following form

$$A_{tt} \frac{\partial^2 \mathcal{Y}}{\partial t^2} = \bar{A}_{20} \frac{\partial^2 \mathcal{Y}}{\partial X^2} + \bar{A}_{11} \frac{\partial^2 \mathcal{Y}}{\partial Y \partial X} + \bar{A}_{02} \frac{\partial^2 \mathcal{Y}}{\partial Y^2} + \bar{A}_{01} \frac{\partial \mathcal{Y}}{\partial Y} + \bar{A}_{10} \frac{\partial \mathcal{Y}}{\partial X}, \quad (9)$$

where the coefficients are

$$\bar{A}_{02} = \frac{1}{4\pi r^4} a_1 a_1'^2 + \mu \pi r^2 \left(\frac{a_1'^2}{4a_1} \cos^2 \theta \right. \quad (10)$$

$$\left. + a_1 e^\Lambda \sin^2 \theta \right), \quad (11)$$

$$\bar{A}_{20} = \mu \pi r^4 \left(\frac{a_1'^2}{4a_1} \sin^2 \theta + \frac{a_1 e^\Lambda \cos^2 \theta}{r^2} \right), \quad (12)$$

$$\bar{A}_{11} = \mu \pi r^4 \left(\frac{a_1'^2}{2a_1} - 2 \frac{a_1 e^\Lambda}{r^2} \right) \sin \theta \cos \theta, \quad (13)$$

$$\bar{A}_{01} = \frac{X}{2\pi r^4} a_1 \left(\frac{2}{r^2} a_1 e^{2\Lambda} - 4\pi j_1 e^{2\Lambda} - \frac{a_1'^2}{2} \right), \quad (14)$$

$$+ \mu' \frac{a_1'^2}{2\sqrt{a_1}} \quad (15)$$

$$+ \frac{\mu \pi r^4}{\sqrt{a_1}} \left(\frac{a_1'^2}{a_1} - 2 \frac{a_1}{r} - \frac{a_1 e^\Lambda}{r^2} - 2\pi J_1 \right) \cos \theta, \quad (16)$$

$$\bar{A}_{10} = \left[\frac{\mu}{\sqrt{a_1}} \left(-\frac{a_1'^2}{4a_1} + 2 \frac{a_1'}{r} - 2\pi J_1 \right) \sin \theta \right. \quad (17)$$

$$\left. - 3\mu \sqrt{a_1} e^\Lambda \frac{\cos \theta}{r^2} + \mu' \frac{a_1'^2}{2\sqrt{a_1}} \cos \theta \right] \pi r^4. \quad (18)$$

Note that for $\mu = 0$ one recovers equation (5). The value of μ in the following is calculated from v_s , the shear velocity, see Schumaker & Thorne (1983)

$$v_s = (\mu/\rho)^{1/2} \simeq 1 \times 10^8 \text{ cm/s}. \quad (19)$$

2.1 Boundary Conditions

In order to solve the equations (5) and (9), appropriate boundary conditions must be imposed. Actually, the boundary conditions in spherical coordinates were reported in Sotani et al. (2007a).

Since the wave equation is given in the (X, Y) coordinates the aforementioned boundary conditions translate as follows:

- regularity at the center: $\mathcal{Y}(X, Y) = 0$;
- no traction on the surface for the open lines:

$$\left[X \frac{\partial \mathcal{Y}}{\partial X} + Y \frac{\partial \mathcal{Y}}{\partial Y} \right] = 0; \quad (20)$$

- axisymmetry

$$Y \frac{\partial \mathcal{Y}}{\partial X} = 0 \quad \text{at} \quad X = 0; \quad (21)$$

- equatorial plane symmetry for $\ell = 2$ initial data i.e.

$$Y \frac{\partial \mathcal{Y}}{\partial X} = 0 \quad \text{at} \quad Y = 0; \quad (22)$$

- equatorial plane symmetry for $\ell = 3$ initial data i.e.

$$\mathcal{Y}(X, Y) = 0 \quad \text{at} \quad Y = 0; \quad (23)$$

- continuity of the radial function a_1 when the sign is switched in the coordinate transformation (4) at $a_1'(r) = 0$.

An additional boundary condition is needed at the base of the fluid-core interface: this condition in spherical coordinates is given by

$$\partial_r \mathcal{Y}_{(-)} = \left[1 + \frac{(2\ell-1)(2\ell+3)}{3(\ell^2+\ell-1)} \frac{v_s^2}{v_A^2} \right] \partial_r \mathcal{Y}_{(+)} \quad (24)$$

where $v_A = B/(4\pi\rho)^{1/2}$ is the Alfvén velocity, see Sotani et al. (2007a) and Glampedakis et al. (2006). In the (X, Y) coordinates the previous condition becomes:

$$\left[X \frac{\partial \mathcal{Y}}{\partial X} + Y \frac{\partial \mathcal{Y}}{\partial Y} \right]_{(-)} = \left[1 + \frac{(2\ell-1)(2\ell+3)}{3(\ell^2+\ell-1)} \frac{v_s^2}{v_A^2} \right] \left[X \frac{\partial \mathcal{Y}}{\partial X} + Y \frac{\partial \mathcal{Y}}{\partial Y} \right]_{(+)}. \quad (25)$$

Although in this two dimensional study we have not used decomposition in spherical harmonics, we kept the relation (25) in order to be able, during the mode recycling, to infer in the best possible way the form of the eigenfunction that corresponds to a specific angular index ℓ . For the initial runs we have used the more generic formula:

$$\partial_r \mathcal{Y}_{(-)} = \left[1 + v_s^2/v_A^2 \right] \partial_r \mathcal{Y}_{(+)} \quad (26)$$

which has been also used by Gabler et al. (2011) to extract the general picture of the spectrum. In general, neither the spectrum nor the eigenfunctions depend critically on this choice.

2.2 Numerical method

We used two representative EOSs the APR (Akmal et al. 1998) and the WFF (Wiringa et al. (1988)), for various mass models and different values of the magnetic field. At the end the ones that seem to fit better the observational data are the ones with a mass of $1.4M_\odot$ while the value for the magnetic field strength which is in accordance to independent estimations was $B_\mu = 4 \times 10^{15}$ Gauss on the pole.

We used a numerical equidistant grid 60×60 in the (X, Y) coordinates, setting $X_{\max} = \sqrt{a_1 \max} \sin \theta$ and $Y_{\max} = \sqrt{a_1 \max} \cos \theta$ and varying θ from 0 to $\pi/2$. The accuracy of the code was tested, performing a simulation with a 90×90 equidistant grid: the results show that the frequencies are not significantly influenced by a change in the number of grid points. The stability of the scheme was also checked: our simulation lasted 2 seconds without showing any signs of instability. The base of crust $(X_{\text{crust}}, Y_{\text{crust}})$ is taken at $\rho = 2.4 \times 10^{14} \text{ g/cm}^3$, according to the crust model by Negele & Vautherin (1973) (NV). In this way the study is divided into two evolution problems coupled via the the interface condition (25).

First, we evolve the 1-dimensional wave equation (5) till the base of the crust $(X_{\text{crust}}, Y_{\text{crust}})$. Special care should

be taken at the point where the coordinate transformation (4) is changing sign, i.e. at $a_{1\text{ max}}$. In practice, one evolves along “magnetic strings” throughout the core of the star till the base of the crust. Then in the crust, we evolve the 2-dimensional wave equation (9), and the oscillating “magnetic strings” of the core are replaced by a “magneto-elastic membrane”.

As a test run for our model, we reproduce the results in Sotani et al. (2007a) for crust oscillations. In this paper, the authors study the torsional oscillations of a non-magnetized star. They also consider, as a first approximation, the no-traction condition in the core-crust interface, i.e. instead of the condition (24) they require that \mathcal{Y} has to be continuous through the interface i.e. $\mathcal{Y}'_{(+)} = \mathcal{Y}'_{(-)}$. Here, we cannot set the magnetic field equal to zero since our coordinate system (X, Y) is based on the magnetic field strength i.e. on $a_1(r)$. However, it was shown in Sotani et al. (2007a) that the influence of the magnetic field on the frequencies becomes important only if $B > 10^{15}$ Gauss, thus we can choose a very low magnetic field for magnetars, e.g. $B = 10^{14}$ Gauss, and perform test runs in order to reproduce the results of Sotani et al. (2007a). Using this magnetic field and the no-traction condition on the interface, we find the results reported in Sotani et al. (2007a). This test verified that the code can reproduce earlier results for the torsional oscillations of the crust.

In a similar way we can isolate the continua found in Colaiuda et al. (2009) and trace its changes due to the presence of the crust. One of the characteristic properties of the continua is the scaling of the frequencies that seem to form the edges of the continua. That is, the edges of the various continua follow the rule that has been found for odd type of initial data in Colaiuda et al. (2009) i.e. that

$$f_{L_n} \approx (n+1)f_{L_0} \quad \text{and} \quad f_{U_n} \approx (n+1)f_{U_0}, \quad (27)$$

where f_L stands for the lower frequency of the continua with an eigenfunction located mainly near the polar axis, and f_U stands for the upper frequency of the continua with maximum amplitude near the last open magnetic field line. These scaling laws for the continua can be easily spotted in Figs. 1 and 2.

In this coupled core-crust problem apart from the continua and the crust modes, we found the imprints of a new family of discrete modes, a detailed description of these modes is presented in the following section.

Finally, we made some additional numerical tests i.e. we set initial data in order to excite only the crust and to study the propagation of the energy in the system. We pick up two points, one in the core and one in the crust, and we follow the evolution of the perturbation during the time of our simulation, performing a FFT at different times and comparing the amplitude of their peaks. We found that the energy quickly flows from the crust towards the core, exciting the Alfvén continua as seen by Levin (2007) and by Gabler et al. (2011). In the same way, the perturbations of the core excite the crust: for this reason it is more appropriate to define the modes generated by the continua, global modes, i.e. modes that involve both crust and core oscillations. The conclusion of this test is in agreement with earlier suggestions (Glampedakis et al. 2006) claiming that when the magnetic field permeates the whole the star, the perturbations cannot be confined in the crust. Another no-

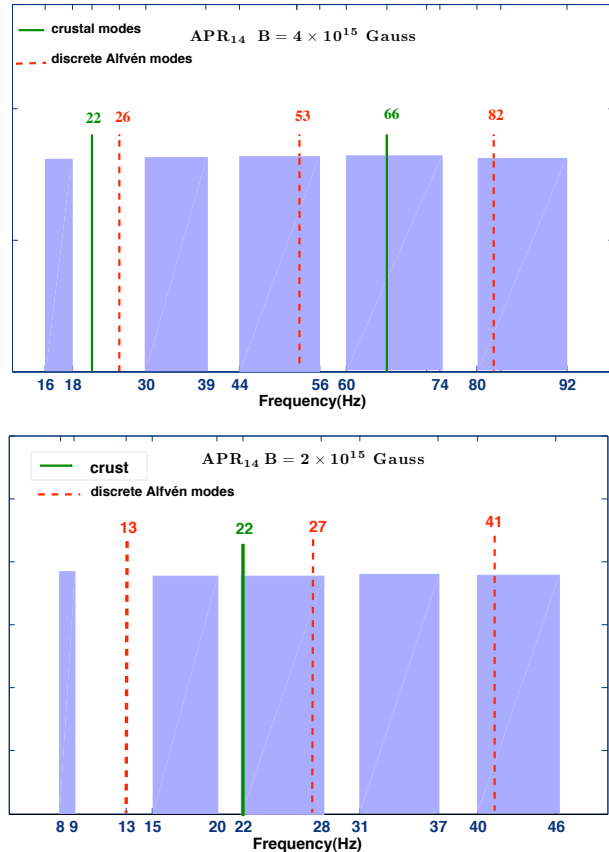


Figure 1. Identification of the frequencies of SGR 1806-20. We show that a stellar model APR with mass $M = 1.4M_{\odot}$ and radius $R = 11.57\text{km}$ can explain all the frequencies observed. *Upper panel:* the magnetic field strength is $4 \times 10^{15}\text{G}$. *Lower panel:* the magnetic field strength is $2 \times 10^{15}\text{G}$. In both panels the blue bands represent the continuous spectrum of the core oscillations.

table result is that even when we associated a mode to the crust, the actual eigenfunction was not confined in the crust but it was extended throughout the star, see Fig. 3. In addition, we find that when a crustal mode is embedded in the continua, its eigenfunction shows the same structure of the continua that hosts it. Contrary, when a crust frequency is found in the gap between the continua, its eigenfunction shows a structure more similar to the one of the fundamental crustal frequencies, compare the upper panel of Fig. 3 and the lower panel of Fig. 4.

3 RESULTS & DISCUSSION

The scope of this paper is not only to understand if the oscillations are global modes (i.e. combined crust-core oscillations), pure crustal modes or pure core modes but also to identify the magnetar models that can fit the observed QPOs. This can potentially lead to a better understanding of the order with which the various modes were excited and left their imprints in the signal. Moreover, a proper identification of the observed QPOs will unveil the details of the magnetar structure.

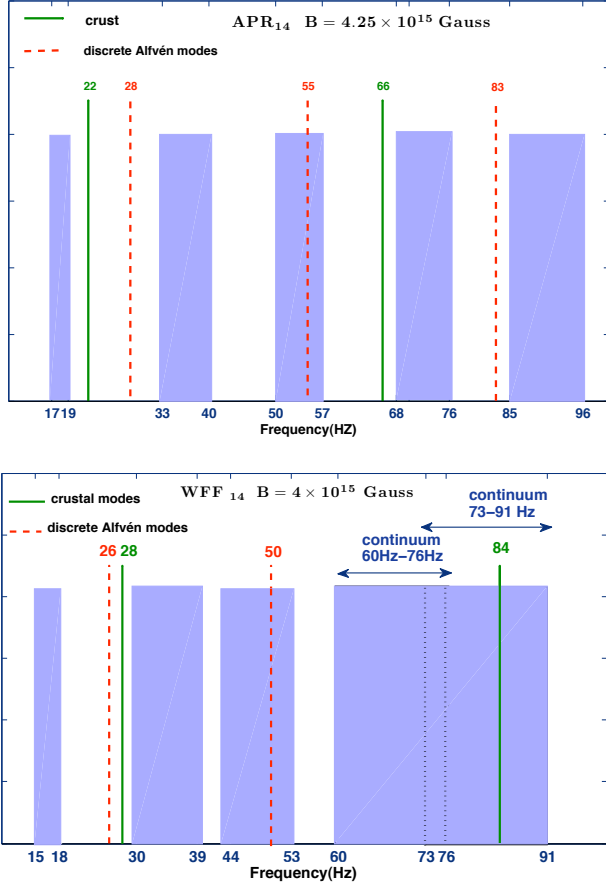


Figure 2. Identification of the frequencies of SGR 1900+14. Since just few frequencies were identified for this SGR, we show that a stellar model APR (*upper panel*) with mass $M = 1.4M_{\odot}$, radius $R = 11.57\text{km}$ and magnetic field $B = 4.25 \times 10^{15}\text{G}$ can explain all the frequencies observed, as well as a stellar model WFF with mass $M = 1.4M_{\odot}$, radius $R = 10.91\text{km}$ and magnetic field $B = 4 \times 10^{15}\text{G}$ (*lower panel*). In both panels the blue bands represent the continuous spectrum of the core oscillations.

3.1 Analysis of the spectrum

As we already pointed out and as it is shown in Fig. 1 and Fig. 2, the spectrum of the oscillations is composed by three different types of modes:

crustal modes: they are associated with the crust and they acquire a discrete spectrum. We found that, when the magnetic field increases, some modes are absorbed by the continua, compare upper panel of Fig. 5, where $B \simeq 10^{14}$ Gauss, with the lower panel of the same figure, where $B \simeq 6 \times 10^{14}$ Gauss: it is clear that some modes disappear. The strong coupling with the crust leads to an effective transfer of energy from the crust to the core, especially when the crust frequency is embedded in the continua (see for example the 66Hz frequency in the upper panel of Fig. 1): from a combined analysis of the peak amplitude of a crust frequency embedded in the continua and of the peak amplitude of the edge of the hosting continua, we found that the energy lost by the crust frequency is stored in the edges of the continua. In addition, we found that the fundamental crustal frequency does not scale with the increase or de-

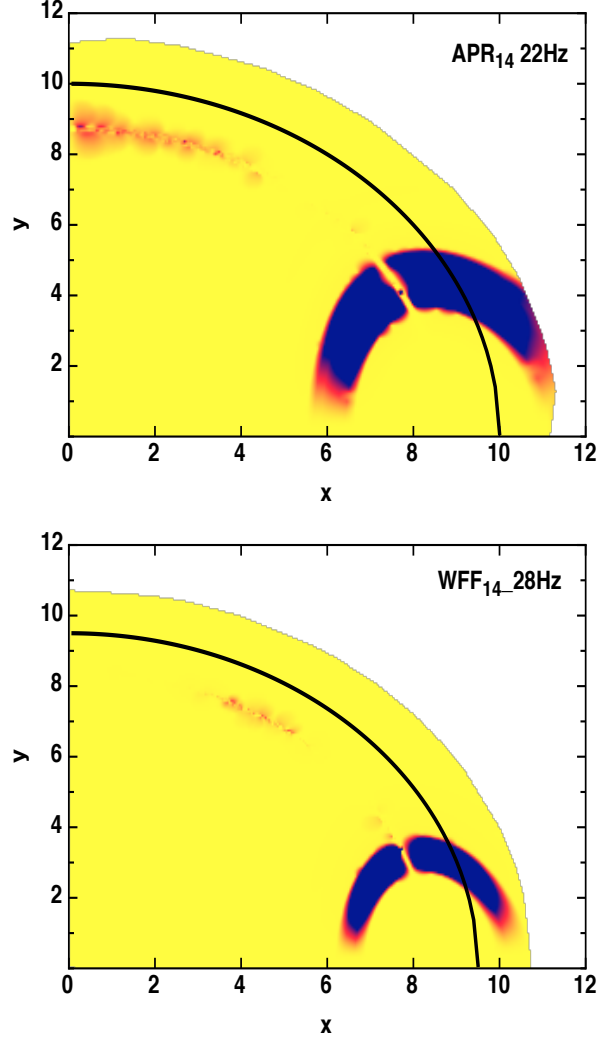


Figure 3. Typical eigenfunctions of a “crust” mode for the model APR₁₄ that identifies the SGR 1806-20 (*upper panel*) and the model WFF₁₄ (*lower panel*), that identifies the SGR 1900+14. This latter frequency is unlikely to be observed, since it does not reach the stellar surface. The noise visible in both figures corresponds to the change of sign of the (X, Y) coordinate at the point $a_{1\text{max}}$.

crease of the magnetic field: this behaviour assures us that the crustal frequencies are not scaling with the magnetic field unlike the Alfvén modes. For magnetic field $B > 10^{15}$ Gauss, the relation among the fundamental frequency and the overtones, is given by:

$$f_{\ell}^{\text{crust}} = (\ell - 1)f_{\ell=2}^{\text{crust}} \quad \text{for } \ell > 2 \quad \text{and } n = 0. \quad (28)$$

For a magnetic field $B < 4 \times 10^{14}$ Gauss, the law found by Samuelsson & Andersson (2007) is recovered:

$$f_{\ell}^{\text{crust}} = \sqrt{(\ell - 1)(\ell + 2)} \frac{v_s}{2\pi R} \quad \text{for } n = 0. \quad (29)$$

For intermediate magnetic fields, a proper scaling cannot be found. The problem of the scaling of the crustal mode was treated also by Gabler et al. (2011), where the authors discuss the appearance or disappearance of the crustal mode in relation to the magnetic field strength.

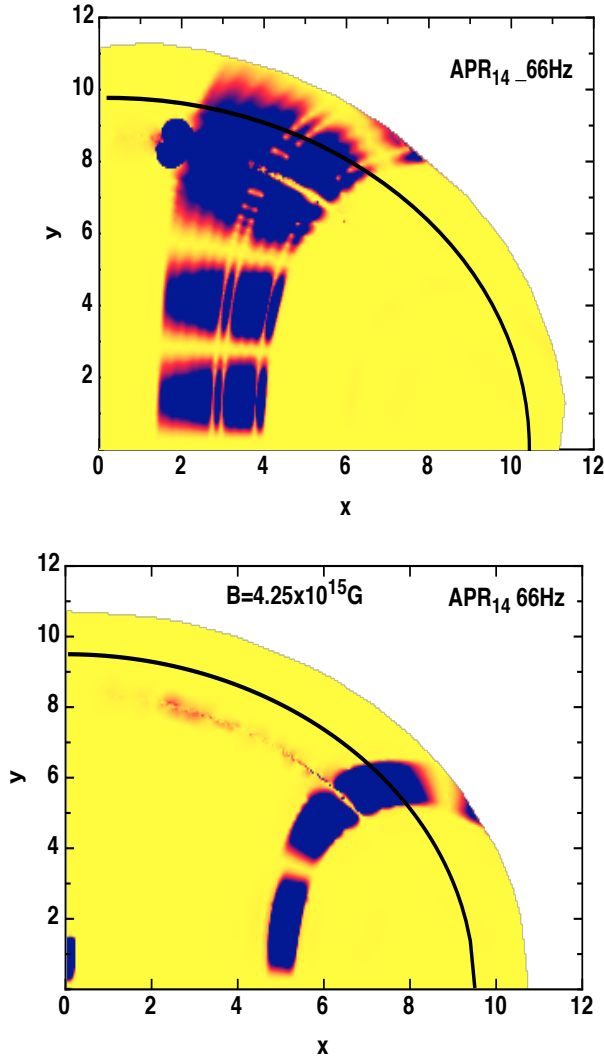


Figure 4. Eigenfunctions of the “crust” mode 66Hz for the APR₁₄. *Upper panel:* the the magnetic field strength is $B = 4 \times 10^{15} \text{G}$ and the frequency is hosted in the continua (model for SGR 1806-20). *Lower panel:* the frequency is located in a gap and the magnetic field is $B = 4.25 \times 10^{15} \text{G}$ (model for SGR 1900+14).

continuous spectrum: the continuous spectrum is generated by the presence of a strong magnetic field ($B > 10^{15}$ Gauss). It is generated in the magnetised fluid present in the core but, in the case of strong magnetic field it extends and, through the coupling with the crust, its oscillations penetrate also in the crust, giving rise to global modes. When the magnetic field increases, the continuous spectrum becomes more and more energetic, absorbing also some crust modes. In addition, the edges of the continua scale with the magnetic field, since $f = B/(\sqrt{4\pi\rho}L)$, where $L = L(X)$ is the length of the “magnetic strings” along which the perturbation propagates (compare the upper and lower panel of Fig. 1). The frequencies of the continua show the scaling described by equation (27). Note that the continuous spectrum always shows gaps in its structure for the model APR (see Fig. 1), while the model WFF does not have a gap between the fourth and fifth continua that overlap (see Fig.

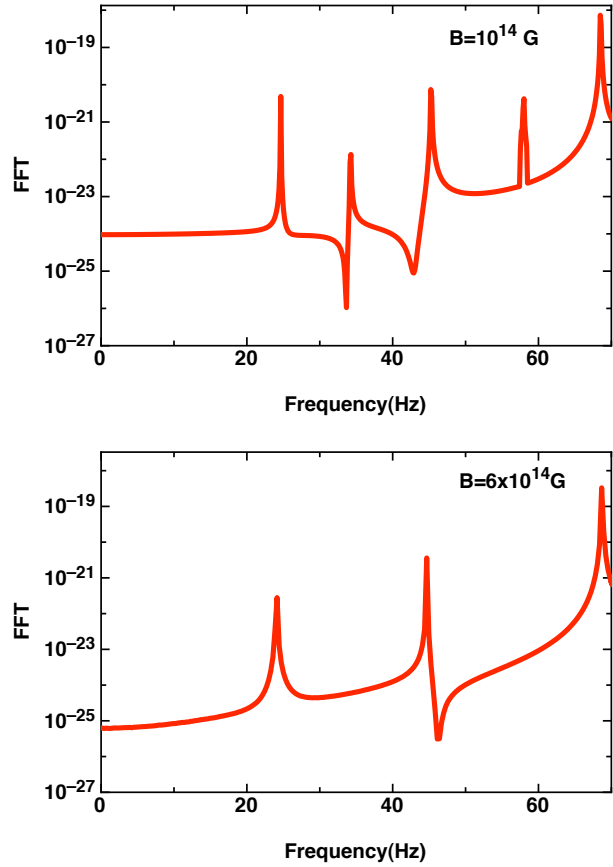


Figure 5. *Upper panel:* the crustal frequencies of an APR₁₄ model for a weak magnetic field $B \simeq 10^{14}$ Gauss. *Lower panel:* for a weak magnetic field $B \simeq 6 \times 10^{14}$ Gauss: some of the frequencies disappear as the magnetic field increases.

2) : the presence/absence of gaps seems then to be a characteristic of a particular model more than a general one, as pointed out in van Hoven & Levin (2011).

discrete Alfvén modes: these modes, as the crustal modes, are discrete but we identify them as Alfvén modes, because they scale with the magnetic field as the frequencies of the continua. Their structure is similar to the one of the crustal mode (compare the upper panel of Fig. 7 and Fig. 8 with Fig. 3) but their scaling follows the scaling observed in the continua:

$$f_n^{(D)} \approx (n+1)f_0^{(D)}. \quad (30)$$

Note that those modes have not been observed in the case of the absence of a crust (see Colaiuda et al. (2009)). For this reason, they can be interpreted as an effect of the coupling between the fluid core and a solid crust (see van Hoven & Levin (2011)).

3.2 Identification of the QPOs

From the variety of magnetars models that we have examined in order to fit to the observational data only a few provide oscillation frequencies that are in agreement with the observed QPOs. More specifically, we compared the data that our numerical code produced with the frequencies from

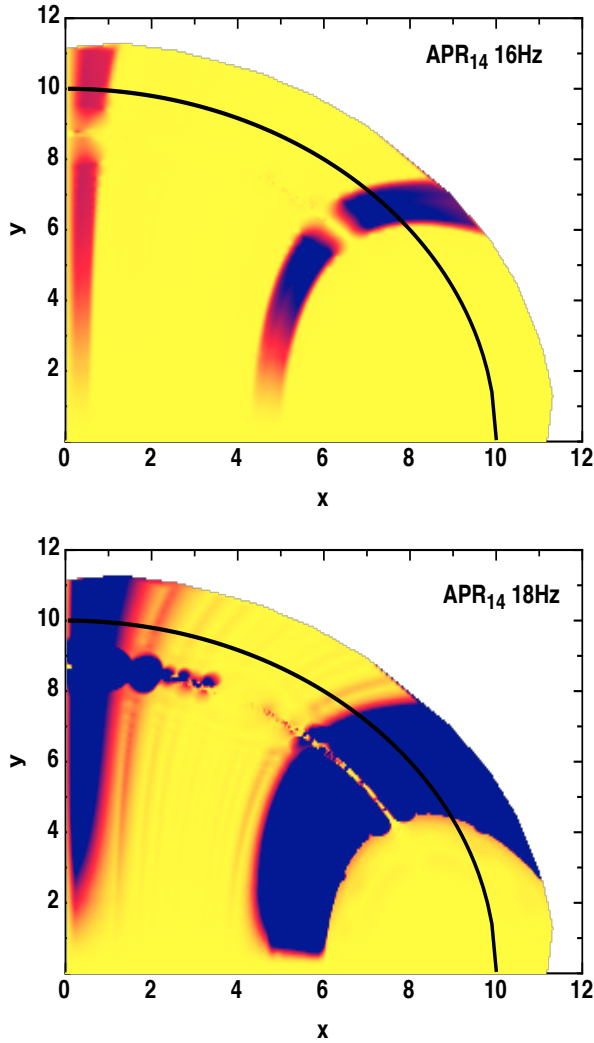


Figure 6. Eigenfunctions of the first two edges of the first continuum for the model APR₁₄ (SGR 1806-20).

the timing analysis of the SGR 1806-20 and SGR 1900+14 shown in Strohmayer & Watts (2006). In this paper, the authors identify several QPOs, of different duration, in the tail of the two events. In particular, for the more recent event, SGR 1806-20, the identified frequencies are 18, 26, 30, 92, 150, 625 and 1840 Hz while for the SGR 1900+14 they found the following frequencies: 28, 53, 84 and 155 Hz. A more recent study by Hambaryan et al. (2011), based on predictions by Colaiuda et al. (2009), confirms, by using a different analysis technique, the earlier results and, in addition, unveils the presence of at least three new frequencies for the SGR 1806-20 these are 16, 21 and 36 Hz.

The new results pose extra challenge, since already the previous studies of pure torsional oscillations in magnetars could not explain all observed frequencies, in particular the lower ones (18, 26, 29 Hz), mainly because of the small spacing among them. If the two new frequencies (16Hz and 21Hz) are added then it is impossible by any means the explanation via crust oscillations alone. On the other hand, the Alfvén continua in the core (Colaiuda et al. 2009; Cerdá-Durán et al. 2009) offer better chances for an explanation

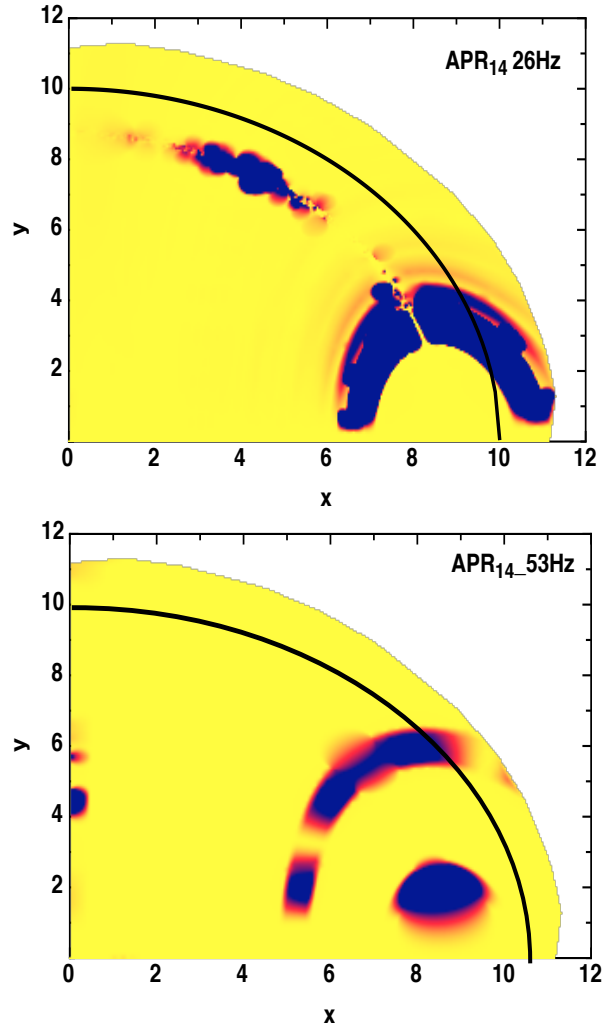


Figure 7. Typical eigenfunctions of a “discrete Alfvén mode” for the model APR₁₄ (SGR 1806-20). Note that the 53Hz frequency, represented in the lower panel, is unlikely to be observed, since it does not reach the stellar surface.

but still both calculations did not take into account the presence of a crust.

Here we show that our modelling of magnetar dynamics can explain all the observed frequencies, and inversely via this explanation we can constrain the parameters of the magnetar (radius, mass, equation of state and magnetic field strength). In Figs. 1 and 2 we graphically show our findings, which comprised by 3 types of oscillations, the Alfvén continua in the core, the discrete Alfvén modes (whose eigenfunctions are plotted in Figs. 7 and 8) and the crustal torsional modes confined mainly in the crust (whose eigenfunctions are plotted in Fig. 3). The eigenfunctions were found using the eigenfunctions recycling program already used in Gaertig & Kokkotas (2009) (see there for more details).

A model with the equation of state (EOS) APR for the core and the NV model for the crust, with mass $M = 1.4M_{\odot}$ and radius $R = 11.57\text{km}$, could fit the new data discovered by Hambaryan et al. (2011) as well as the previous data for the SGR 1806-20 (Watts & Strohmayer 2007). Here, we fo-

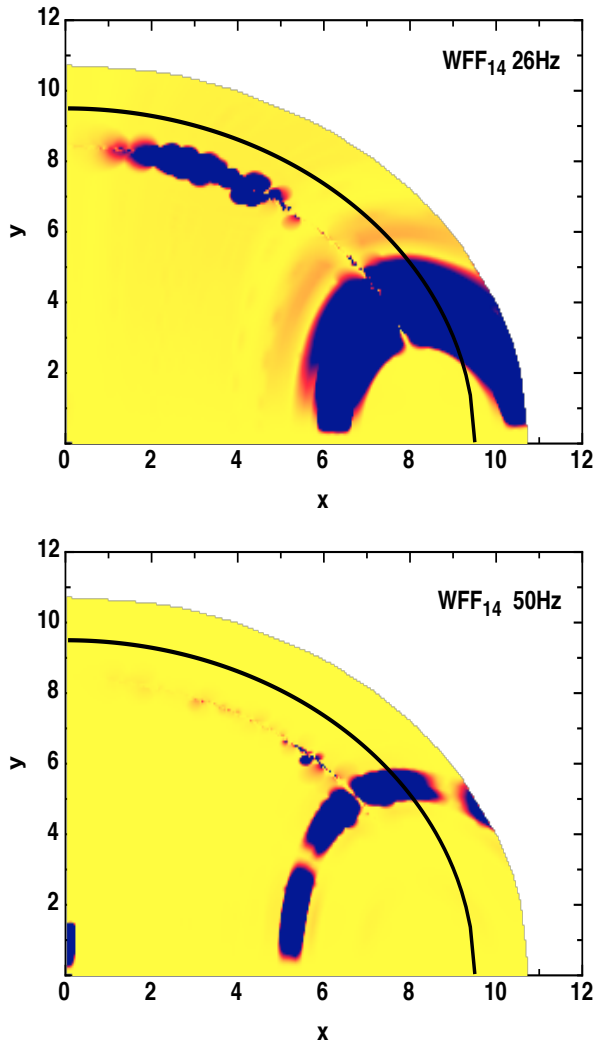


Figure 8. Typical eigenfunctions of a “discrete Alfvén mode” for the model WFF₁₄ (SGR 1900+14).

cus on the first five frequencies, i.e 16, 18, 22, 26, 29 Hz, since they are the more difficult to be explained, the higher ones can be explained in various ways, as multiples of the lower ones or as polar modes (see Sotani & Kokkotas (2009)).

In the upper panel of Fig. 6, the eigenfunction of the 16Hz frequency, i.e. the first edge of the continua (see Fig. 1), is shown. The perturbation involves both core and crust and it can be classified as a global oscillation, with the crust and the core oscillating together due to the strong coupling induced by the magnetic field. The perturbation seems to be localised near the magnetic axis and just outside the region with the closed lines. In the bottom panel of Fig. 6, the eigenfunction of the second edge of the first continuum, the 18Hz frequency, is shown: its structure is similar to the one of the 16Hz frequency, but it shows broader spreading and intense oscillation.

The 22 Hz mode has a discrete nature, as we already pointed out in the previous section, see upper panel in Fig. 3. Studying the time evolution of the perturbation, as explained in the previous section, we find that the core is excited after some time, when the crust successfully transfers

its energy to it. It seems to be a fossil of crust mode found for weaker magnetic field and it is then reasonable to classify it as crustal frequency. The next frequency found, at 26 Hz, see Fig. 7, is similar in structure to the crustal one but, analysing its time evolution, we find that it originates at the crust core interface and its nature is discrete. In addition, although it is a discrete frequency, it scales with the magnetic field: it has then all the characteristics of a discrete Alfvén mode.

Finally, at a frequency of ≈ 30 Hz, we observe excitation mainly of the core region but we notice also significant excitation in the crust and thus this frequency can be identified as a global oscillation, that is localised mainly in the core and then forces the crust to oscillate violently.

Note that the higher frequencies are located closer to the magnetic axis than the lower ones. This behaviour was already seen in Colaiuda et al. (2009) and Cerdá-Durán et al. (2009). Note also that the part of the star that seems not to be excited corresponds to the closed magnetic field lines. As we already noted in Colaiuda et al. (2009), the closed magnetic lines have a significant smaller amplitude than the open magnetic field lines and cannot be observed via oscillations or thermal surface phenomena, since they are confined in the interior of the star.

Concerning the SGR 1900+14, an unique model that can fit better the observed frequencies cannot be found because of the paucity of the observations. For this reason, we find that two models can fit the identified QPOs for the SGR 1900+14: the EOS APR with a mass $M = 1.4M_{\odot}$, radius $R = 11.57\text{km}$ and a magnetic field strength $B = 4.25 \times 10^{15}\text{G}$, as well as the EOS WFF with mass and radius respectively $M = 1.4 M_{\odot}$ and $R = 10.91\text{km}$, and a magnetic field strength $B = 4 \times 10^{15}\text{G}$. We focus our study on the identification of the first three frequencies observed: 28Hz, 53Hz and 84Hz. In the case of APR₁₄ with a magnetic field strength $B = 4.25 \times 10^{15}\text{G}$, all the three frequencies can be identified as discrete Alfvén modes, see upper panel of Fig. 2. The oscillations involve both crust and core. Contrary, in the case of model WFF₁₄ (see lower panel of Fig. 2), the frequencies can be identified as global modes (the 28Hz and the 54Hz) and as crustal mode (the 84Hz). In particular, the 28Hz frequency is identified as the first edge of the second continuum while the 54Hz is identified as the second edge of the third continuum (see Fig. 2): both those frequencies are global modes, with excitations that involve both the crust and the core. The 84Hz is a crust frequency: its nature is discrete and the oscillations, although initially confined in the crust, expand rapidly in the core. Also in this model, we find a discrete Alfvén mode around 26Hz.

Note that the equation of state used for the crust in all the models that we present in this paper is the NV. As already pointed out in Sotani et al. (2007b), the use of the EOS proposed by Douchin & Haensel (2001) for the crust does not alter significantly the results. Still since it produces a thinner crust, it creates difficulties in the simulations since it demands finer grid in the crust in order to evolve properly the perturbations inside it.

The results present in this section partially agree with the ones found by Lee (2008), where a similar problem of a star with solid crust and fluid core both permeated by a magnetic field is studied in Newtonian theory as an eigenvalue problem. Without time evolutions only discrete Alfvén fre-

quencies could be found and thus no continuous spectrum appears. However, the discrete Alfvén frequencies that we found here agree qualitatively with the ones described by Lee.

It is worth mentioning that we analysed also alternative combination of magnetic field configuration, consisting of both poloidal and toroidal magnetic fields. We found that the presence of a toroidal component in the background magnetic field shifts the spectrum towards lower values. This extra parameter complicates significantly the procedure of identifying the magnetar model that will better fit the observed frequencies.

Note that a more accurate treatment of the shear modulus than the one used in equation (19) may change quantitative the crustal modes. In addition, the presence of a superfluid component in the inner crust and/or in the core could affect the frequencies significantly, as discussed in Andersson et al. (2009).

4 CONCLUSIONS

We studied the torsional oscillations of a magnetar in a general relativistic framework, consisting of a relativistic neutron star with a solid crust and a fluid core. The core and the crust are coupled by the strong magnetic field and this coupling is defined by the appropriate boundary conditions. We find that the presence of a crust and its coupling with the core partially alters the earlier picture presented in Colaiuda et al. (2009).

In particular, the presence of a crust makes the spectrum denser and thus offers an explanation of the nature of all the observed frequencies. We can distinguish, using the eigenfunction, between global modes (i.e. modes for which both the crust and the core oscillate), crust modes and discrete Alfvén modes (i.e. modes that have a discrete nature but that scale with the magnetic field). Unlike the paper by van Hoven & Levin (2011), we find that also in the case of intermediate magnetic field ($B < 10^{15}$ G), the oscillations is not confined to the core but also the crust is excited. We find a family of modes, the discrete Alfvén modes, in the gaps between two contiguous continua (as found by van Hoven & Levin (2011)) and, in addition, we can resolve them also inside the continua.

We can identify all the frequencies observed in SGR 1806-20 and SGR 1900+14 and this allows us to put some constraints on radius, mass, crust thickness and magnetic field strength. In particular, the frequencies observed in SGR 1806-20 can be fitted uniquely with the APR stellar model with a mass $M = 1.4 M_{\odot}$, a radius $R = 11.57$ km, a compactness $M/R = 0.178$ and a crust thickness $\Delta r/R = 0.099$. The magnetic field strength at the pole is $B = 4 \times 10^{15}$ G. Contrary, the SGR 1900+14 cannot be strongly constrained because of the limited information from the observations, i.e. just three QPOs were identified. For this reason, the SGR 1900+14 cannot be reproduced uniquely by a single model, as it has been done for the SGR 1806-20, since at least two stellar models could reproduce its observed frequencies with a good accuracy: the APR stellar model with a mass $M = 1.4 M_{\odot}$, a radius $R = 11.57$ km, a compactness $M/R = 0.178$ and a crust thickness $\Delta r/R = 0.099$ but with a magnetic field strength $B = 4.25 \times 10^{15}$ G and the WFF

model with a mass $M = 1.4 M_{\odot}$, a radius $R = 10.91$ km, a compactness $M/R = 0.189$, crust thickness $\Delta r/R = 0.085$ and a magnetic field strength $B = 4 \times 10^{15}$ G. Note that, in all the models, the crust is described by the NV equation of state.

ACKNOWLEDGMENTS

This work was supported by the German Science Council (DFG) via SFB/TR7. We thank Erich Gaertig for providing us his recycling routine. We are grateful to the anonymous referee, to Yuri Levin, to Maarten van Hoven and to Sandro Mereghetti for their useful suggestions and comments.

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